Gauge invariance in gravity-like descriptions of massive gauge field theories

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We discuss gravity-like formulations of massive Abelian and non-Abelian gauge field theories in four space-time dimensions with particular emphasis on the issue of gauge invariance. Alternative descriptions in terms of antisymmetric tensor fields and geometric variables, respectively, are analysed. In both approaches Stückelberg degrees of freedom factor out. We also demonstrate, in the Abelian case, that the massless limit for the gauge propagator, which does not exist in the vector potential formulation, is well-defined for the antisymmetric tensor fields.

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I. INTRODUCTION

Massive gauge bosons belong to the fundamental concepts we use for picturing nature. Prominent examples are found in the physics of electroweak interactions, superconductivity, and confinement. Even more than in the massless case, gauge invariance is a severe constraint for the construction of massive gauge field theories. Usually additional fields beyond the original gauge field have to be included in order to obtain gauge invariant expressions.[26] Technically, this is linked to the fact that the aforementioned gauge field—the Yang–Mills connection—changes inhomogeneously under gauge transformations and encodes also spurious degrees of freedom arising from the construction principle of gauge invariance. This complicates the extraction of physical quantities. A variety of approaches has been developed in order to deal with this situation. Wilson loops [1] represent gauge invariant but non-local variables.[27] Alternatively, there exist decomposition techniques like the one due to Cho, Faddeev, and Niemi [2]. Here we first pursue a reformulation of massive Yang–Mills theories in terms of antisymmetric gauge algebra valued tensor fields $B^a_{\mu\nu}$ (Sect. II) and subsequently continue with a representation in terms of geometric variables (Sect. III).

In Sect. II A we review the massless case. It is related to gravity [3] formulated as BF gravity [4] and thus linked to quantum gravity. The antisymmetric tensor field can be seen as dual field strength and transforms homogeneously under gauge transformations. This fact already makes it simpler to keep track of gauge invariance. In Sect. II B the generalisation to the massive case is presented. In the $B^a_{\mu\nu}$ field representation the (non-Abelian) Stückelberg fields, which are commonly present in massive gauge field theories and needed there in order to keep track of gauge invariance, factor out completely. In other words, no scalar fields are needed for a gauge invariant formulation of massive gauge field theories in terms of antisymmetric tensor fields. The case of a constant mass is linked to sigma models (gauged and ungauged) in different respects. Sect. II B 1 contains the generalisation to a position dependent mass, which corresponds to introducing the Higgs degree of freedom. In Sect. II B 2 non-diagonal mass terms are admitted. This is necessary to accommodate the Weinberg–Salam model, which is studied as particular case.

Sect. III presents a description of the massive case, with constant and varying mass, in terms of geometric variables. In this step the remaining gauge degrees of freedom are eliminated. The emergent description is in terms of local colour singlet variables. Finally, Sect. III A is concerned with the geometric representation of the Weinberg–Salam model.

The Appendix treats the Abelian case. It allows to better interpret and understand several of the findings in the non-Abelian settings. Of course, in the Abelian case already the $B_{\mu\nu}$ field is gauge invariant. Among other things, we demonstrate that the $m \to 0$ limit of the gauge propagator for the $B_{\mu\nu}$ fields is well-defined as opposed to the ill-defined limit for the A_μ field propagator.

Sect. IV summarises the paper.

II. ANTISYMMETRIC TENSOR FIELDS

A. Massless

Before we investigate massive gauge field theories let us recall some details about the massless case. The partition function of a massless non-Abelian gauge field theory without fermions is given by

$$P := \int [dA] \exp\{i\textstyle\int d^4x\mathcal{L}\}, \tag{1}$$

with the Lagrangian density

$$\mathcal{L} = \mathcal{L}_0 := -\tfrac{1}{4g^2} F^a_{\mu\nu} F^{a\mu\nu} \tag{2}$$

and the field tensor

$$F^a_{\mu\nu} := \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + f^{abc} A^b_\mu A^c_\nu. \tag{3}$$

A^a_μ stands for the gauge field, f^{abc} for the antisymmetric structure constant, and g for the coupling constant. [28] Variation of the classical action with respect to the gauge field gives the classical Yang–Mills equations

$$D^{ab}_\mu(A) F^{b\mu\nu} = 0, \tag{4}$$

where the covariant derivative is defined as $D^{ab}_\mu(A) := \delta^{ab}\partial_\mu + f^{acb}A^c_\mu$. The partition function in the first-order formalism can be obtained after multiplying Eq. (1) with a prefactor in form of a Gaussian integral over an antisymmetric tensor field $B^a_{\mu\nu}$,

$$P \;\cong\; \int [dA][dB]\exp\{i\textstyle\int d^4x[\mathcal{L}_0 - \frac{g^2}{4}B^a_{\mu\nu}B^{a\mu\nu}]\}. \quad (5)$$

("$\cong$" indicates that in the last step the normalisation of the partition function has been changed.) Subsequently, the field $B^a_{\mu\nu}$ is shifted by $\frac{1}{g^2}\tilde{F}^a_{\mu\nu}$, where the dual field tensor is defined as $\tilde{F}^a_{\mu\nu} := \frac{1}{2}\epsilon_{\mu\nu\kappa\lambda}F^{a\kappa\lambda}$,

$$\begin{aligned} P \;=\; & \int [dA][dB]\times \\ & \times\exp\{i\textstyle\int d^4x[-\frac{1}{2}\tilde{F}^a_{\mu\nu}B^{a\mu\nu} - \frac{g^2}{4}B^a_{\mu\nu}B^{a\mu\nu}]\}. \quad (6)\end{aligned}$$

In this form the partition function is formulated in terms of the Yang–Mills connection A^a_μ and the antisymmetric tensor field $B^a_{\mu\nu}$ as independent variables. Variation of the classical action with respect to these variables leads to the classical equations of motion

$$g^2 B^a_{\mu\nu} = -\tilde{F}^a_{\mu\nu} \quad \text{and} \quad D^{ab}_\mu(A)\tilde{B}^{b\mu\nu} = 0, \quad (7)$$

where $\tilde{B}^a_{\mu\nu} := \frac{1}{2}\epsilon_{\kappa\lambda\mu\nu}B^{a\kappa\lambda}$. By eliminating $B^a_{\mu\nu}$ the original Yang–Mills equation (4) is reproduced. Every term in the classical action in the partition function (6) contains at most one derivative as opposed to two in Eq. (1). This explains the name "first-order" formalism. The classical action in Eq. (6) is invariant under simultaneous gauge transformations of the independent variables according to

$$\begin{aligned} A^{a\mu}T^a \;&=:\; A^\mu \to A^\mu_U := U[A^\mu - iU^\dagger(\partial^\mu U)]U^\dagger \quad (8)\\ B^{a\mu\nu}T^a \;&=:\; B^{\mu\nu} \to B^{\mu\nu}_U := UB^{\mu\nu}U^\dagger, \quad (9)\end{aligned}$$

or infinitesimally,

$$\begin{aligned} \delta A^a_\mu \;&=\; \partial_\mu\theta^a + f^{abc}A^b_\mu\theta^c \\ \delta B^a_{\mu\nu} \;&=\; f^{abc}B^b_{\mu\nu}\theta^c. \quad (10)\end{aligned}$$

The T^a stand for the generators of the gauge group. From the Bianchi identity $D^{ab}_\mu(A)\tilde{F}^{b\mu\nu} = 0$ follows a second symmetry of the BF term alone: Infinitesimally, for unchanged A^a_μ,

$$\delta B^a_{\mu\nu} = \partial_\mu\vartheta^a_\nu - \partial_\nu\vartheta^a_\mu + f^{abc}(A^b_\mu\vartheta^c_\nu - A^b_\nu\vartheta^c_\mu). \quad (11)$$

A particular combination of the transformations (10) and (11), $\theta^a = n^\mu A^a_\mu$ and $\vartheta^a_\nu = n^\mu B^a_{\mu\nu}$, corresponds to the transformation of a tensor under an infinitesimal local coordinate transformation $x^\mu \to x^\mu - n^\mu(x)$,

$$\delta B_{\mu\nu} = B_{\lambda\nu}\partial_\mu n^\lambda + B_{\mu\lambda}\partial_\nu n^\lambda + n^\lambda\partial_\lambda B_{\mu\nu}, \quad (12)$$

that is a diffeomorphism. Hence, the BF term is diffeomorphism invariant, which explains why this theory is also known as BF gravity. The BB term is not diffeomorphism invariant and, hence, imposes a constraint. The combination of the two terms amounts to an action of Plebanski type which are studied in the context of quantum gravity [3, 4].

We now would like to eliminate the Yang–Mills connection by integrating it out. For fixed $B^a_{\mu\nu}$ the integrand of the path integral is not gauge invariant with respect to gauge transformations of the gauge field A^a_μ alone; the field tensor $F^a_{\mu\nu}$ transforms homogeneously and the corresponding gauge transformations are not absorbed if $B^a_{\mu\nu}$ is held fixed. Therefore, the integral over the gauge group is in general not cyclic which otherwise would render the path integral ill-defined. The term in the exponent linear in the gauge field A^a_μ, $A^a_\nu\partial_\mu\tilde{B}^{a\mu\nu}$, is obtained by carrying out a partial integration in which surface terms are ignored. Afterwards it is absorbed by shifting A^a_μ by $(\mathbb{B}^{-1})^{ab}_{\mu\nu}(\partial_\lambda\tilde{B}^{b\lambda\nu})$, where $\mathbb{B}^{ab}_{\mu\nu} := f^{abc}\tilde{B}^a_{\mu\nu}$. In general its inverse $(\mathbb{B}^{-1})^{ab}_{\mu\nu}$, defined by $(\mathbb{B}^{-1})^{ab}_{\mu\nu}\mathbb{B}^{bc}_{\kappa\lambda}g^{\nu\kappa} = \delta^{ac}g_{\mu\lambda}$ exists in three or more space-time dimensions [5]. We are left with a Gaussian integral in A^a_μ giving the inverse square-root of the determinant of $\mathbb{B}^{ab}_{\mu\nu}$,

$$\begin{aligned} \mathrm{Det}^{-\frac{1}{2}}\mathbb{B} \;&:=\; \prod_x \det{}^{-\frac{1}{2}}\mathbb{B} \cong \\ &\cong\; \int [da]\exp\{-\tfrac{i}{2}\textstyle\int d^4x a^{a\mu}\mathbb{B}^{ab}_{\mu\nu}a^{b\nu}\}. \quad (13)\end{aligned}$$

In the last expression $\mathbb{B}^{ab}_{\mu\nu}$ appears in the place of an inverse gluon propagator, that is sandwiched between two gauge fields. This analogy carries even further: Interpreting $\partial_\mu\tilde{B}^{a\mu\nu}$ as a current, $(\mathbb{B}^{-1})^{ab}_{\mu\nu}(\partial_\lambda\tilde{B}^{b\lambda\nu})$, the current together with the "propagator" $(\mathbb{B}^{-1})^{ab}_{\mu\nu}$, is exactly the abovementioned term to be absorbed in the gauge field A^a_μ. Finally, we obtain,

$$\begin{aligned} P \;\cong\; & \int [dB]\mathrm{Det}^{-\frac{1}{2}}\mathbb{B}\exp\{i\textstyle\int d^4x[-\frac{g^2}{4}B^a_{\mu\nu}B^{a\mu\nu} - \\ & -\tfrac{1}{2}(\partial_\kappa\tilde{B}^{a\kappa\mu})(\mathbb{B}^{-1})^{ab}_{\mu\nu}(\partial_\lambda\tilde{B}^{b\lambda\nu})]\}. \quad (14)\end{aligned}$$

This result is known from [5, 6, 7]. The exponent in the previous expression corresponds to the value of the $[dA]$ integral at the saddle-point value $\breve{A}^a_\mu$ of the gauge field. It obeys the classical field equation (7). Using $\breve{A}^a_\mu(B) = (\mathbb{B}^{-1})^{ab}_{\mu\nu}(\partial_\lambda\tilde{B}^{b\lambda\nu})$ the second term in the above exponent can be rewritten as $-\frac{i}{2}\int d^4x\tilde{B}^a_{\mu\nu}F^{a\mu\nu}[\breve{A}(B)]$, which involves an integration by parts and makes its gauge invariance manifest. The fluctuations a^a_μ around the saddle point $\breve{A}^a_\mu$, contributing to the partition function (6), are Gaussian because the action in the first-order formalism is only of second order in the gauge field A^a_μ. They give rise to the determinant (13). What happens if a zero of the determinant is encountered can be understood by looking at the Abelian case discussed in Appendix A. There the BF term does not fix a gauge for the integration over the gauge field A_μ because the Abelian field tensor $F^{\mu\nu}$ is gauge invariant. If it is performed

nevertheless one encounters a functional δ distribution which enforces the vanishing of the current $\partial_\mu \tilde{B}^{\mu\nu}$. In this sense the zeros of the determinant in the non-Abelian case arise if $\tilde{B}^a_{\mu\nu}$ is such that the BF term does not totally fix a gauge for the $[dA]$ integration, but leaves behind a residual gauge invariance. It in turn corresponds to vanishing components of the current $\partial_\mu \tilde{B}^{a\mu\nu}$. (Technically, there then is at least one flat direction in the otherwise Gaussian integrand. The flat directions are along those eigenvectors of $\mathbb{B}$ possessing zero eigenvalues.)

When incorporated with the exponent, which requires a regularisation [8], the determinant contributes a term proportional to $\frac{1}{2}\ln\det\mathbb{B}$ to the action. This term together with the $\tilde{B}B$ term constitutes the effective potential, which is obtained from the exponent in the partition function after dropping all terms containing derivatives of fields. The effective potential becomes singular for field configurations for which $\det\mathbb{B} = 0$. It is gauge invariant because all contributing addends are gauge invariant separately.

The classical equations of motion obtained by varying the action in Eq. (14) with respect to the dual antisymmetric tensor field $\tilde{B}^{a\mu\nu}$ are given by

$$\begin{aligned} g^2 \tilde{B}^a_{\mu\nu} &= (g^\rho_\nu g^\sigma_\mu - g^\rho_\mu g^\sigma_\nu)\partial_\rho(\mathbb{B}^{-1})^{ab}_{\sigma\kappa}(\partial_\lambda \tilde{B}^{b\lambda\kappa}) - \\ &\quad -(\partial_\rho \tilde{B}^{d\rho\kappa})(\mathbb{B}^{-1})^{db}_{\kappa\mu} f^{abc}(\mathbb{B}^{-1})^{ce}_{\nu\lambda}(\partial_\sigma \tilde{B}^{e\sigma\lambda}), \end{aligned} \tag{15}$$

which coincides with the first of Eqs. (7) with the field tensor evaluated at the saddle point of the action, $F^a_{\mu\nu}[\breve{A}(B)]$. Taking into account additionally the effect due to fluctuations of A^a_μ contributes a term proportional to $\frac{\delta \mathrm{Det}\mathbb{B}}{\delta \tilde{B}^{a\mu\nu}} \det^{-1}\mathbb{B}$ to the previous equation.

B. Massive

In the massive case the prototypical Lagrangian is of the form $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_m$, where $\mathcal{L}_m := \frac{m^2}{2} A^a_\mu A^{a\mu}$. (Due to our conventions the physical mass is given by $m_{\text{phys}} := mg$.) This contribution to the Lagrangian is of course not gauge invariant. Putting it, regardlessly, into the partition function, gives

$$P = \int [dA][dU] \exp\{i{\textstyle\int} d^4x[\mathcal{L}_0 + \tfrac{m^2}{2}A^a_\mu A^{a\mu}]\}, \tag{16}$$

which can be interpreted as the unitary gauge representation of an extended theory. In order to see this let us split the functional integral over A^a_μ into an integral over the gauge group $[dU]$ and gauge inequivalent field configurations $[dA]'$. Usually this separation is carried out by fixing a gauge according to

$$\int [dA]' := \int [dA]\delta[f^a(A) - C^a]\Delta_f(A). \tag{17}$$

$f^a(A) = C^a$ is the gauge condition and $\Delta_f(A)$ stands for the Faddeev–Popov determinant defined through $1 \stackrel{!}{=} \int [dA]\delta[f^a(A) - C^a]\Delta_f(A)$ [29]. Introducing this reparametrisation into the partition function (16) yields,

$$\begin{aligned} P &= \int [dA]'[dU] \exp(i{\textstyle\int} d^4x\{-\tfrac{1}{4g^2}F^a_{\mu\nu}F^{a\mu\nu} + \\ &\quad + \tfrac{m^2}{2}[A_\mu - iU^\dagger(\partial_\mu U)]^a[A^\mu - iU^\dagger(\partial^\mu U)]^a\}). \end{aligned} \tag{18}$$

$\mathcal{L}_0$ is gauge invariant in any case and remains thus unaffected. In the mass term the gauge transformations appear explicitly [9]. We now replace all of these gauge transformations with an auxiliary (gauge group valued) scalar field Φ, $U^\dagger \to \Phi$, obeying the constraint

$$\Phi^\dagger\Phi \stackrel{!}{\equiv} \mathbb{1}. \tag{19}$$

The field Φ can be expressed as $\Phi =: e^{-i\theta}$, where $\theta =: \theta^a T^a$ is the gauge algebra valued non-Abelian generalisation of the Stückelberg field [10]. For a massive gauge theory they are a manifestation of the longitudinal degrees of freedom of the gauge bosons. In the context of symmetry breaking they arise as Goldstone modes ("pions"). In the context of the Thirring model these observations have been made in [11]. There it was noted as well that the θ is also the field used in the canonical Hamiltonian Batalin–Fradkin–Vilkovisky formalism [12]. We can extract the manifestly gauge invariant classical Lagrangian

$$\mathcal{L}_{\text{cl}} := -\tfrac{1}{4g^2}F^a_{\mu\nu}F^{a\mu\nu} + m^2\mathrm{tr}[(D_\mu\Phi)^\dagger(D^\mu\Phi)], \tag{20}$$

where the scalars have been rearranged making use of the product rule of differentiation and the cyclic property of the trace and where $D_\mu\Phi := \partial_\mu\Phi - iA_\mu\Phi$. Eq. (20) resembles the Lagrangian density of a non-linear gauged sigma model. In the Abelian case the fields θ decouple from the dynamics. For non-Abelian gauge groups they do not and one would have to deal with the non-polynomial coupling to them.

In the following we show that these spurious degrees of freedom can be absorbed when making the transition to a formulation based on the antisymmetric tensor field $B^a_{\mu\nu}$. Introducing the antisymmetric tensor field into the corresponding partition function, like in the previous section, results in,

$$\begin{aligned} P &\cong \int [dA][d\Phi][dB] \exp(i{\textstyle\int} d^4x \times \\ &\quad \times\{-\tfrac{g^2}{4}B^a_{\mu\nu}B^{a\mu\nu} - \tfrac{1}{2}\tilde{F}^a_{\mu\nu}B^{a\mu\nu} + \\ &\quad + \tfrac{m^2}{2}[A_\mu - i\Phi(\partial_\mu\Phi^\dagger)]^a[A^\mu - i\Phi(\partial^\mu\Phi^\dagger)]^a\}). \end{aligned} \tag{21}$$

Removing the gauge scalars Φ from the mass term by a gauge transformation of the gauge field A^a_μ makes them explicit in the BF term,

$$\begin{aligned} P &= \int [dA][d\Phi][dB] \exp\{i{\textstyle\int} d^4x[-\tfrac{g^2}{4}B^a_{\mu\nu}B^{a\mu\nu} - \\ &\quad -\mathrm{tr}(\Phi\tilde{F}_{\mu\nu}\Phi^\dagger B^{\mu\nu}) + \tfrac{m^2}{2}A^a_\mu A^{a\mu}]\}. \end{aligned} \tag{22}$$

In the next step we would like to integrate over the Yang–Mills connection A^a_μ. Already in the previous expression, however, we can perceive that the final result will only depend on the combination of fields $\Phi^\dagger B_{\mu\nu}\Phi$. [The Φ field can also be made explicit in the BB term in form of the constraint (19).] Therefore, the functional integral over Φ only covers multiple times the range which is already covered by the $[dB]$ integration. Hence the degrees of freedom of the field Φ have become obsolete in this formulation and the $[d\Phi]$ integral can be factored out. Thus, we could have performed the unitary gauge calculation right from the start. In either case, the final result reads,

$$\begin{aligned} P &\cong \int [dB] \mathrm{Det}^{-\frac{1}{2}} \mathbb{M} \exp\{i\textstyle\int d^4x[-\frac{g^2}{4}B^a_{\mu\nu}B^{a\mu\nu} - \\ &\quad -\tfrac{1}{2}(\partial_\kappa \tilde{B}^{a\kappa\mu})(\mathbb{M}^{-1})^{ab}_{\mu\nu}(\partial_\lambda \tilde{B}^{b\lambda\nu})]\}, \qquad (23)\end{aligned}$$

where $\mathbb{M}^{ab}_{\mu\nu} := \mathbb{B}^{ab}_{\mu\nu} - m^2\delta^{ab}g_{\mu\nu}$, which coincides with [13]. $\mathbb{M}^{ab}_{\mu\nu}$ and hence $(\mathbb{M}^{-1})^{ab}_{\mu\nu}$ transform homogeneously under the adjoint representation. In Eq. (14) the central matrix $(\mathbb{B}^{-1})^{ab}_{\mu\nu}$ in the analogous term transformed in exactly the same way. There this behaviour ensured the gauge invariance of this term's contribution to the classical action. Consequently, the classical action in the massive case has the same invariance properties. In particular, the aforementioned gauge invariant classical action describes a massive gauge theory without having to resort to additional scalar fields. For $\det\mathbb{B} \neq 0$, the limit $m \to 0$ is smooth. For $\det\mathbb{B} = 0$ the conserved current components alluded to above would have to be separated appropriately in order to recover the corresponding δ distributions present in these situations in the massless case.

Again the effective action is dominated by the term proportional to $\frac{1}{2}\det\mathbb{M}$. The contribution from the mass to $\mathbb{M}$ shifts the eigenvalues from the values obtained for $\mathbb{B}$. Hence the singular contributions are typically obtained for eigenvalues of $\mathbb{B}$ of the order of m^2. The effective potential is again gauge invariant, for the same reason as in the massless case.

The classical equations of motion obtained by variation of the action in Eq. (21) are given by,

$$\begin{aligned} g^2 B^a_{\mu\nu} &= -\tilde{F}^a_{\mu\nu}, \\ D^{ab}_\mu(A)\tilde{B}^{b\mu\nu} &= -m^2[A^\nu - i\Phi(\partial^\nu\Phi^\dagger)]^a, \\ 0 &= \tfrac{\delta}{\delta\theta^b}\textstyle\int d^4x\{[A^a_\mu - i\Phi(\partial_\mu\Phi^\dagger)]^a\}^2. \qquad (24)\end{aligned}$$

In these equations a unique solution can be chosen, that is a gauge be fixed, by selecting the scalar field Φ. $\Phi \equiv 1$ gives the unitary gauge, in which the last of the above equations drops out. The general non-Abelian case is difficult to handle already on the classical level, which is one of the main motivations to look for an alternative formulation. In the non-Abelian case, the equation of motion obtained from Eq. (23) resembles strongly the massless case,

$$\begin{aligned} g^2\tilde{B}^a_{\mu\nu} &= (g^\rho_\nu g^\sigma_\mu - g^\rho_\mu g^\sigma_\nu)\partial_\rho(\mathbb{M}^{-1})^{ab}_{\sigma\kappa}(\partial_\lambda\tilde{B}^{b\lambda\kappa}) - \\ &\quad -(\partial_\rho\tilde{B}^{d\rho\kappa})(\mathbb{M}^{-1})^{db}_{\kappa\mu}f^{abc}(\mathbb{M}^{-1})^{ce}_{\nu\lambda}(\partial_\sigma\tilde{B}^{e\sigma\lambda}), \qquad (25)\end{aligned}$$

insofar as all occurrences of $(\mathbb{B}^{-1})^{ab}_{\mu\nu}$ have been replaced by $(\mathbb{M}^{-1})^{ab}_{\mu\nu}$. Incorporation of the effect of the Gaussian fluctuations of the gauge field A^a_μ would give rise to a contribution proportional to $\frac{\delta\mathbb{B}}{\delta\tilde{B}^{a\mu\nu}}\det^{-1}\mathbb{M}$ in the previous equation.

Before we go over to more general cases of massive non-Abelian gauge field theories, let us have a look at the weak coupling limit: There the BB term in Eq. (21) is neglected. Subsequently, integrating out the $B^a_{\mu\nu}$ field enforces $F^a_{\mu\nu} \equiv 0$. [This condition also arises from the classical equation of motion (24) for g=0.] Hence, for vanishing coupling exclusively pure gauge configurations of the gauge field A^a_μ contribute. They can be combined with the Φ fields and one is left with a non-linear realisation of a partition function,

$$P \underset{(21)}{\overset{g=0}{\cong}} \int [d\Phi]\exp\{im^2\textstyle\int d^4x\ \mathrm{tr}[(\partial_\mu\Phi^\dagger)(\partial^\mu\Phi)]\}, \qquad (26)$$

of a free massless scalar [13]. Setting $g = 0$ interchanges with integrating out the $B^a_{\mu\nu}$ field from the partition function (21). Thus, the partition function (23) with $g = 0$ is equivalent to (26). That a scalar degree of freedom can be described by means of an antisymmetric tensor field has been noticed in [14].

1. *Position-dependent mass and the Higgs*

One possible generalisation of the above set-up is obtained by softening the constraint (19). This can be seen as allowing for a position dependent mass. The new degree of freedom ultimately corresponds to the Higgs. When introducing the mass m as new degree of freedom (as "mass scalar") we can restrict its variation by introducing a potential term $V(m^2)$, which remains to be specified, and a kinetic term $K(m)$, which we choose in its canonical form $K(m) = \frac{1}{2}(\partial_\mu m)(\partial^\mu m)$. It gives a penalty for fast variations of m between neighbouring space-time points. The fixed mass model is obtained in the limit of an infinitely sharp potential with its minimum located at a non-zero value for the mass. Putting together the partition function in unitary gauge leads to,

$$\begin{aligned} P &= \int [dA][dm]\exp\{i\textstyle\int d^4x[-\frac{1}{4g^2}F^a_{\mu\nu}F^{a\mu\nu} + \\ &\quad +\tfrac{m^2}{2\mathcal{N}}A^a_\mu A^{a\mu} + K(m) + V(m^2)]\}, \qquad (27)\end{aligned}$$

where we have introduced the normalisation constant $\mathcal{N} := \dim \mathrm{R}$, with R standing for the representation of the scalars. This factor allows us to keep the canonical normalisation of the mass scalar m. We can now repeat

the same steps as in the previous section in order to identify the classical Lagrangian,

$$\mathcal{L}_{\rm cl} := -\tfrac{1}{4g^2}F^a_{\mu\nu}F^{a\mu\nu} + \mathcal{N}^{-1}{\rm tr}[(D_\mu\phi)^\dagger(D^\mu\phi)] + V(|\phi|^2),$$

where now $\phi := m\Phi$. In order to reformulate the partition function in terms of the antisymmetric tensor field we can once more repeat the steps in the previous section. Again the spurious degrees of freedom represented by the field Φ can be factored out. Finally, this gives [15],

$$\begin{aligned} P \;\cong\; & \int [dB][dm]{\rm Det}^{-\frac{1}{2}}\mathbb{M}\exp\{i\textstyle\int d^4x[-\frac{g^2}{4}B^a_{\mu\nu}B^{a\mu\nu} - \\ & -\tfrac{1}{2}(\partial_\kappa\tilde{B}^{a\kappa\mu})(\mathbb{M}^{-1})^{ab}_{\mu\nu}(\partial_\lambda\tilde{B}^{b\lambda\nu}) + \\ & +K(m) + V(m^2)]\}, \qquad (28)\end{aligned}$$

where $\mathbb{M}^{ab}_{\mu\nu} = \mathbb{B}^{ab}_{\mu\nu} - m^2\mathcal{N}^{-1}\delta^b g_{\mu\nu}$ depends on the spacetime dependent mass m. The determinant can as usual be included with the exponent in form of a term proportional to $\frac{1}{2}\det\mathbb{M}$, the pole of which will dominate the effective potential. As just mentioned, however, $\mathbb{M}$ is also a function of m. Hence, in order to find the minimum, the effective potential must also be varied with respect to the mass m.

Carrying the representation in terms of antisymmetric tensor fields another step further, the partition function containing the kinetic term $K(m)$ of the mass scalar can be expressed as Abelian version of Eq. (26),

$$\begin{aligned} & \int [db][da]\exp\{i\textstyle\int d^4x[-\frac{1}{2}\tilde{b}_{\mu\nu}f^{\mu\nu} + \frac{1}{2}a_\mu a^\mu]\} = \\ = & \int [dm]\exp\{i\textstyle\int d^4x[\frac{1}{2}(\partial_\mu m)(\partial^\mu m)]\}, \qquad (29)\end{aligned}$$

where here the mass scalar m is identified with the Abelian gauge parameter. Combining the last equation with the partition function (28) all occurrences of the mass scalar m can be replaced by the phase integral $m \to \int dx^\mu a_\mu$. The bf term enforces the curvature f to vanish which constrains a_μ to pure gauges $\partial_\mu m$ and the aforementioned integral becomes path-independent.

2. Non-diagonal mass term and the Weinberg–Salam model

The mass terms investigated so far had in common that all the bosonic degrees of freedom they described possessed the same mass. A more general mass term would be given by $\mathcal{L}_m := \frac{m^2}{2}A^a_\mu A^{b\mu}\mathbf{m}^{ab}$. Another similar approach is based on the Lagrangian $\mathcal{L}_m := \frac{m^2}{2}{\rm tr}\{A_\mu A^\mu\Psi\}$ where Ψ is group valued and constant. We shall begin our discussion with this second variant and limit ourselves to a Ψ with real entries and ${\rm tr}\Psi = 1$, which, in fact, does not impose additional constraints. Using this expression in the partition function (27) and making explicit the gauge scalars yields,

$$\begin{aligned} P \;=\; & \int [dA][dm]\exp\{i\textstyle\int d^4x[-\frac{1}{4g^2}F^a_{\mu\nu}F^{a\mu\nu} + \\ & +\tfrac{1}{2}{\rm tr}\{(D_\mu\phi)^\dagger(D^\mu\phi)\Psi\} + V(m^2)]\}. \qquad (30)\end{aligned}$$

Expressed in terms of the antisymmetric tensor field $B^a_{\mu\nu}$, the corresponding partition function coincides with Eq. (28) but with $\mathbb{M}^{ab}_{\mu\nu}$ replaced by $\mathbb{M}^{ab}_{\mu\nu} := \mathbb{B}^{ab}_{\mu\nu} - m^2{\rm tr}\{T^aT^b\Psi\}g_{\mu\nu}$.

Let us now consider directly the $SU(2)\times U(1)$ Weinberg–Salam model. Its partition function can be expressed as,

$$\begin{aligned} P \;=\; & \int [dA][d\psi]\exp\{i\textstyle\int d^4x[-\frac{1}{4g^2}F^{\underline{a}}_{\mu\nu}F^{\underline{a}\mu\nu} + \\ & +\tfrac{1}{2}\psi^\dagger(\overleftarrow{\partial}_\mu + iA_\mu)(\overrightarrow{\partial}^\mu - iA^\mu)\psi + V(|\psi|^2)]\}, \\ & \qquad (31)\end{aligned}$$

where ψ is a complex scalar doublet, $A_\mu := A^{\underline{a}}_\mu T^{\underline{a}}$, with $\underline{a} \in \{0;\ldots;3\}$, T^a here stands for the generators of $SU(2)$ in fundamental representation, and, accordingly, T^0 for $\frac{g_0}{2g}$ times the 2×2 unit matrix, with the $U(1)$ coupling constant g_0. The partition function can be reparametrised with $\psi = m\Phi\hat{\psi}$, where $m = \sqrt{|\psi|^2}$, Φ is a group valued scalar field as above, and $\hat{\psi}$ is a constant doublet with $|\hat{\psi}|^2 = 1$. The partition function then becomes,

$$\begin{aligned} P \;=\; & \int [dA][d\Phi][dm]\exp(i\textstyle\int d^4x\{-\frac{1}{4g^2}F^{\underline{a}}_{\mu\nu}F^{\underline{a}\mu\nu} + \\ & +\tfrac{m^2}{2}{\rm tr}[\Phi^\dagger(\overleftarrow{\partial}_\mu + iA_\mu)(\overrightarrow{\partial}^\mu - iA^\mu)\Phi\Psi] + \\ & +\tfrac{1}{2}(\partial_\mu m)(\partial^\mu m) + V(m^2)\}), \qquad (32)\end{aligned}$$

where

$$\Psi = \hat{\psi}\otimes\hat{\psi}^\dagger. \qquad (33)$$

Making the transition to the first order formalism leads to

$$\begin{aligned} P \;\cong\; & \int [dA][dB][d\Phi][dm]\exp(i\textstyle\int d^4x\{-\frac{g^2}{4}B^{\underline{a}}_{\mu\nu}B^{\underline{a}\mu\nu} - \\ & -\tfrac{1}{2}F^{\underline{a}}_{\mu\nu}\tilde{B}^{\underline{a}\mu\nu} + K(m) + V(m^2) + \\ & +\tfrac{m^2}{2}{\rm tr}[\Phi^\dagger(\overleftarrow{\partial}_\mu + iA_\mu)(\overrightarrow{\partial}^\mu - iA^\mu)\Phi\Psi]\}). \qquad (34)\end{aligned}$$

As in the previous case, a gauge transformation of the gauge field A^a_μ can remove the gauge scalar Φ from the mass term (despite the matrix Ψ). Thereafter Φ only appears in the combination $\Phi^\dagger B_{\mu\nu}\Phi$ and the integral $[d\Phi]$ merely leads to repetitions of the $[dB]$ integral. [The U(1) part drops out completely right away.] Therefore the $[d\Phi]$ integration can be factored out,

$$\begin{aligned} P \;\cong\; & \int [dA][dB][dm]\exp\{i\textstyle\int d^4x[-\frac{g^2}{4}B^{\underline{a}}_{\mu\nu}B^{\underline{a}\mu\nu} - \\ & -\tfrac{1}{2}F^{\underline{a}}_{\mu\nu}\tilde{B}^{\underline{a}\mu\nu} + \tfrac{m^2}{2}{\rm tr}(A_\mu A^\mu\Psi) + \\ & +K(m) + V(m^2)]\}. \qquad (35)\end{aligned}$$

The subsequent integration over the gauge fields $A^{\underline{a}}_\mu$ leads to

$$\begin{aligned} P &\cong \int [dB][dm]\mathrm{Det}^{-\frac{1}{2}}\mathbb{M}\exp\{i\int d^4x[-\tfrac{g^2}{4}B^{\underline{a}}_{\mu\nu}B^{\underline{a}\mu\nu} - \\ &\quad -\tfrac{1}{2}(\partial_\kappa \tilde{B}^{\underline{a}\kappa\mu})(\mathbb{M}^{-1})^{\underline{ab}}_{\mu\nu}(\partial_\lambda \tilde{B}^{\underline{b}\lambda\nu}) + \\ &\quad + K(m) + V(m^2)]\}, \end{aligned} \tag{36}$$

where $\mathbb{M}^{\underline{ab}}_{\mu\nu} := \mathbb{B}^{\underline{ab}}_{\mu\nu} - m^2\mathrm{tr}(T^{\underline{a}}T^{\underline{b}}\Psi)g_{\mu\nu}$.

From hereon we continue our discussion based on the mass matrix

$$\mathbf{m}^{\underline{ab}} := \tfrac{1}{2}\mathrm{tr}(\{T^{\underline{a}}, T^{\underline{b}}\}\Psi), \tag{37}$$

which had already been mentioned at the beginning of Sect. II B 2. $\mathbf{m}^{\underline{ab}}$ is real and has been chosen to be symmetric. (Antisymmetric parts are projected out by the contraction with the symmetric $A^{\underline{a}}_\mu A^{\underline{b}\mu}$.) Thus it possesses a complete orthonormal set of eigenvectors $\mu^{\underline{b}}_j$ with the associated real eigenvalues m_j, $\mathbf{m}^{\underline{ab}}\mu^{\underline{b}}_j \stackrel{!}{=} \not\Sigma_j m_j \mu^{\underline{a}}_j$. With the help of these normalised eigenvectors one can construct projectors $\pi^{\underline{ab}}_j := \not\Sigma_j \mu^{\underline{a}}_j \mu^{\underline{b}}_j$ and decompose the mass matrix, $\mathbf{m}^{\underline{ab}} = m_j \pi^{\underline{ab}}_j$. The projectors are complete, $\mathbb{1}^{\underline{ab}} = \Sigma_j \pi^{\underline{ab}}_j$, idempotent $\not\Sigma_j \pi^{\underline{ab}}_j \pi^{\underline{bc}}_j = \pi^{\underline{ac}}_j$, and satisfy $\pi^{\underline{ab}}_j \pi^{\underline{bc}}_k \stackrel{j\neq k}{=} 0$. The matrix $\mathbb{B}^{\underline{ab}}_{\mu\nu}$, the antisymmetric tensor field $B^a_{\mu\nu}$, and the gauge field A^a_μ can also be decomposed with the help of the eigenvectors: $\mathbb{B}^{\underline{ab}}_{\mu\nu} = \mu^{\underline{a}}_j \mathbb{b}^{jk}_{\mu\nu}\mu^{\underline{b}}_k$, where $\mathbb{b}^{jk}_{\mu\nu} := \mu^{\underline{a}}_j \mathbb{B}^{\underline{ab}}_{\mu\nu}\mu^{\underline{b}}_k$; $B^{\underline{a}}_{\mu\nu} = b^j_{\mu\nu}\mu^{\underline{a}}_j$, where $b^j_{\mu\nu} := B^{\underline{a}}_{\mu\nu}\mu^{\underline{a}}_j$; and $A^{\underline{a}}_\mu = a^j_\mu \mu^{\underline{a}}_j$, where $a^j_\mu := A^{\underline{a}}_\mu \mu^{\underline{a}}_j$. Using this decomposition in the partition function (36) leads to,

$$\begin{aligned} P &\cong \int [db][dm]\mathrm{Det}^{-\frac{1}{2}}\mathbb{m}\exp\{i\int d^4x[-\tfrac{g^2}{4}b^j_{\mu\nu}b^{j\mu\nu} - \\ &\quad -\tfrac{1}{2}(\partial_\kappa \tilde{b}^{j\kappa\mu})(\mathbb{m}^{-1})^{jk}_{\mu\nu}(\partial_\lambda \tilde{b}^{k\lambda\nu}) + \\ &\quad + K(m) + V(m^2)]\}, \end{aligned} \tag{38}$$

where $\mathbb{m}^{jk}_{\mu\nu} := \mathbb{b}^{jk}_{\mu\nu} - m^2\sum_l m_l \delta^{jl}\delta^{kl}g_{\mu\nu}$.

Making use of the concrete form of $\mathbf{m}^{\underline{ab}}$ given in Eq. (37), inserting Ψ from Eq. (33), and subsequent diagonalisation leads to the eigenvalues 0, $\frac{1}{4}$, $\frac{1}{4}$ and $\frac{1}{4}(1+\frac{g_0^2}{g^2})$. These correspond to the photon, the two W bosons and the heavier Z boson, respectively. The thus obtained tree-level Z to W mass ratio squared consistently reproduces the cosine of the Weinberg angle in terms of the coupling constants, $\cos^2\vartheta_w = \frac{g^2}{g^2+g_0^2}$. Due to the masslessness of the photon one addend in the sum over l in the expression $\mathbb{m}^{jk}_{\mu\nu}$ above does not contribute. Still, the total $\mathbb{m}^{jk}_{\mu\nu}$ does not vanish like in the case of a single massless Abelian gauge boson (see Appendix A). Physically this corresponds to the coupling of the photon to the W and Z bosons.

III. GEOMETRIC REPRESENTATION

The fact that the antisymmetric tensor field $B^a_{\mu\nu}$ transforms homogeneously represents already an advantage over the formulation in terms of the inhomogeneously transforming gauge fields A^a_μ. Still, $B^a_{\mu\nu}$ contains degrees of freedom linked to the gauge transformations (9). These can be eliminated by making the transition to a formulation in terms of geometric variables. In this section we provide a classically equivalent description of the massive gauge field theories in terms of geometric variables in Euclidean space for two colours by adapting Ref. [16] to include mass. The first-order action is quadratic in the gauge-field A^a_μ.[30] Thus the evaluation of the classical action at the saddle point yields the expression equivalent to the different exponents obtained after integrating out the gauge field A^a_μ in the various partition functions in the previous section. In Euclidean space the classical massive Yang–Mills action in the first order formalism reads

$$S := \int d^4x(\mathcal{L}_{BB} + \mathcal{L}_{BF} + \mathcal{L}_{AA}), \tag{39}$$

where

$$\mathcal{L}_{BB} = -\tfrac{g^2}{4}B^a_{\mu\nu}B^a_{\mu\nu}, \tag{40}$$

$$\mathcal{L}_{BF} = +\tfrac{i}{4}\epsilon^{\mu\nu\kappa\lambda}B^a_{\mu\nu}F^a_{\kappa\lambda}, \tag{41}$$

$$\mathcal{L}_{AA} = -\tfrac{m^2}{2}A^a_\mu A^a_\mu. \tag{42}$$

At first we will investigate the situation for the unitary gauge mass term $\mathcal{L}_{AA}$ and study the role played by the scalars Φ afterwards.

As starting point it is important to note that a metric can be constructed that makes the tensor $B^a_{\mu\nu}$ self-dual [7]. In order to exploit this fact, it is convenient to define the antisymmetric tensor ($j \in \{1;2;3\}$)

$$T^j_{\mu\nu} := \eta^j_{AB}e^A_\mu e^B_\nu, \tag{43}$$

with the self-dual 't Hooft symbol η^j_{AB} [17] [31] and the tetrad e^A_μ. From there we construct a metric $\mathrm{g}_{\mu\nu}$ in terms of the tensor $T^j_{\mu\nu}$

$$\mathrm{g}_{\mu\nu} \equiv e^A_\mu e^A_\nu = \tfrac{1}{6}\epsilon^{jkl}T^j_{\mu\kappa}T^{k\kappa\lambda}T^l_{\lambda\nu}, \tag{44}$$

where

$$T^{j\mu\nu} := \tfrac{1}{2\sqrt{\mathrm{g}}}\epsilon^{\mu\nu\kappa\lambda}T^j_{\kappa\lambda} \tag{45}$$

and

$$\begin{aligned} (\sqrt{\mathrm{g}})^3 &:= \tfrac{1}{48}(\epsilon_{jkl}T^j_{\mu_1\nu_1}T^k_{\mu_2\nu_2}T^l_{\mu_3\nu_3}) \times \\ &\quad \times(\epsilon_{j'k'l'}T^{j'}_{\kappa_1\lambda_1}T^{k'}_{\kappa_2\lambda_2}T^{l'}_{\kappa_3\lambda_3}) \times \\ &\quad \times \epsilon^{\mu_1\nu_1\kappa_1\lambda_1}\epsilon^{\mu_2\nu_2\kappa_2\lambda_2}\epsilon^{\mu_3\nu_3\kappa_3\lambda_3} \end{aligned} \tag{46}$$

Subsequently, we introduce a triad d^a_j such that

$$B^a_{\mu\nu} =: d^a_j T^j_{\mu\nu}. \tag{47}$$

This permits us to reexpress the BB term of the classical Lagrangian,

$$\mathcal{L}_{BB} = -\tfrac{g^2}{4} T^j_{\mu\nu} h_{jk} T^k_{\mu\nu}, \tag{48}$$

where $h_{jk} := d^a_j d^a_k$. Putting Eqs. (47) and (45) into the saddle point condition

$$\tfrac{1}{2}\epsilon^{\kappa\lambda\mu\nu} D^{ab}_\mu(\breve{A}) B^b_{\kappa\lambda} = +im^2 \breve{A}^a_\nu \tag{49}$$

gives

$$D^{ab}_\mu(\breve{A})(\sqrt{\mathrm{g}} d^b_j T^{j\mu\nu}) = +im^2 \breve{A}^a_\nu. \tag{50}$$

In the following we define the connection coefficients $\gamma_\mu|^k_j$ as expansion parameters of the covariant derivative of the triads at the saddle point in terms of the triads,

$$D^{ab}_\mu(\breve{A}) d^b_j =: \gamma_\mu|^k_j d^a_k. \tag{51}$$

This would not be directly possible for more than two colours, as then the set of triads is not complete. The connection coefficients allow us to define covariant derivatives according to

$$\nabla_\mu|^k_j := \partial_\mu \delta^k_j + \gamma_\mu|^k_j. \tag{52}$$

These, in turn, permit us to rewrite the saddle point condition (49) as

$$d^a_k \nabla_\mu|^k_j(\sqrt{\mathrm{g}} T^{j\mu\nu}) = im^2 \breve{A}^a_\nu, \tag{53}$$

and the mass term in the classical Lagrangian becomes

$$\mathcal{L}_{AA} = \tfrac{1}{2m^2}[\nabla_\mu|^k_i(\sqrt{\mathrm{g}} T^{i\mu\nu})] h_{kl} [\nabla_\kappa|^l_j(\sqrt{\mathrm{g}} T^{j\kappa\nu})]. \tag{54}$$

In the limit $m \to 0$ this term enforces the covariant conservation condition $\nabla_\mu|^k_i(\sqrt{\mathrm{g}} T^{i\mu\nu}) \equiv 0$, known for the massless case. It results also directly from the saddle point condition (53). Here $d^a_k \nabla_\mu|^k_i(\sqrt{\mathrm{g}} T^{i\mu\nu})$ are the direct analogues of the Abelian currents $\epsilon^{\mu\nu\kappa\lambda}\partial_\mu B_{\kappa\lambda}$, which are conserved in the massless case [see Eq. (A6)] and distributed following a Gaussian distribution in the massive case [see Eq. (A10)].

The commutator of the above covariant derivatives yields a Riemann-like tensor $R^k{}_{j\mu\nu}$

$$R^k{}_{j\mu\nu} := [\nabla_\mu, \nabla_\nu]^k_j. \tag{55}$$

By evaluating, in adjoint representation (marked by $^\circ$), the following difference of double commutators $[\mathring{D}_\mu(\breve{A}), [\mathring{D}_\nu(\breve{A}), \mathring{d}_j]] - (\mu \leftrightarrow \nu)$ in two different ways, one can show that

$$i[\mathring{d}_j, \mathring{F}_{\mu\nu}(\breve{A})] = \mathring{d}_k R^k{}_{j\mu\nu}, \tag{56}$$

or in components,

$$F^a_{\mu\nu}(\breve{A}) = \tfrac{1}{2}\epsilon^{abc} d^{bj} d^c_k R^k{}_{j\mu\nu}, \tag{57}$$

where $d^{aj} d^a_k := \delta^j_k$ defines the inverse triad, $d^{aj} = h^{jk} d^a_k$. Hence, we are now in the position to rewrite the remaining BF term of the Lagrangian density. Introducing Eqs. (47) and (57) into Eq. (41) results in

$$\mathcal{L}_{BF} = \tfrac{i}{4}\sqrt{\mathrm{g}} T^{j\mu\nu} R^k{}_{l\mu\nu}\epsilon_{jmk} h^{lm}. \tag{58}$$

Let us now repeat the previous steps with a mass term in which the gauge scalars Φ are explicit,

$$\mathcal{L}^\Phi_{AA} := -\frac{m^2}{2}[A_\mu - i\Phi(\partial_\mu\Phi^\dagger)]^a [A_\mu - i\Phi(\partial_\mu\Phi^\dagger)]^a. \tag{59}$$

In that case the saddle point condition (49) is given by,

$$\tfrac{1}{2}\epsilon^{\kappa\lambda\mu\nu} D^{ab}_\mu(\breve{A}) \tilde{B}^b_{\kappa\lambda} = im^2[\breve{A}_\nu - i\Phi(\partial_\nu\Phi^\dagger)]^a, \tag{60}$$

or in the form of Eq. (53), that is with the left-hand side replaced,

$$d^a_k \nabla_\mu|^k_j(\sqrt{\mathrm{g}} T^{j\mu\nu}) = im^2[\breve{A}_\nu - i\Phi(\partial_\nu\Phi^\dagger)]^a. \tag{61}$$

Reexpressing $\mathcal{L}^\Phi_{AA}$ with the help of the previous equation reproduces exactly the unitary gauge result (54) for the mass term.

Finally, the tensor $\mathbb{B}$ appearing in the determinant (13), which accounts for the Gaussian fluctuations of the gauge field A^a_μ, formulated in the new variables reads $\mathbb{B}^{bc}_{\mu\nu} = \sqrt{\mathrm{g}} f^{abc} d^a_i T^{i\mu\nu}$. Now all ingredients are known which are needed to express the equivalent of the partition function (16) in terms of the new variables. For a position-dependent mass the discussion does not change materially. The potential and kinematic term for the mass scalar m have to be added to the action.

Contrary to the massless case the A^a_μ dependent part of the Euclidean action is genuinely complex. Without mass only the T-odd and hence purely imaginary BF term was A^a_μ dependent. With mass there contributes the additional T-even and thus real mass term. Therefore the saddle point value $\breve{A}^a_\mu$ for the gauge field becomes complex. This is a known phenomenon and in this context it is essential to deform the integration contour of the path integral in the partition function to run through the saddle point [18]. For the Gaussian integrals which are under consideration here, in doing so, we do not receive additional contributions. The imaginary part $\mathcal{I}\breve{A}^a_\mu$ of the saddle point value of the gauge field transforms homogeneously under gauge transformations. The complex valued saddle point of the gauge field which is integrated out does not affect the real-valuedness of the remaining fields, here $B^a_{\mu\nu}$. In this sense the field $B^a_{\mu\nu}$ represents a parameter for the integration over A^a_μ. The tensor $T^j_{\mu\nu}$ is real-valued by definition and therefore the same holds also for the triad d^a_j [see Eq. (47)]. h_{kl} is composed of the triads and, consequently, real-valued as well. The imaginary part of the saddle point value of the gauge field, $\mathcal{I}\breve{A}^a_\mu$, enters the connection coefficients (51). Through them it affects the covariant derivative (52) and

the Riemann-like tensor (55). More concretely the connection coefficients $\gamma_\mu|^k_j$ can be decomposed according to

$$D^{ab}_\mu(\mathcal{R}\breve{A})d^b_j = (\mathcal{R}\gamma_\mu|^k_j)d^a_k, \quad (62)$$
$$f^{abc}(\mathcal{I}\breve{A}^c_\mu)d^b_j = (\mathcal{I}\gamma_\mu|^k_j)d^a_k, \quad (63)$$

with the obvious consequences for the covariant derivative,

$$\nabla_\mu|^k_j = \mathcal{R}\nabla_\mu|^k_j + i\mathcal{I}\nabla_\mu|^k_j, \quad (64)$$
$$\mathcal{R}\nabla_\mu|^k_j = \partial_\mu\delta^k_j + \mathcal{R}\gamma_\mu|^k_j, \quad (65)$$
$$\mathcal{I}\nabla_\mu|^k_j = \mathcal{I}\gamma_\mu|^k_j. \quad (66)$$

This composition reflects in the mass term,

$$\begin{aligned}\mathcal{R}\mathcal{L}_{AA} &= \tfrac{1}{2m^2}\{[\mathcal{R}\nabla_\mu|^k_i(\sqrt{g}T^{i\mu\nu})]h_{kl}[\mathcal{R}\nabla_\kappa|^l_j(\sqrt{g}T^{j\kappa\nu})] - \\ &\quad -[\mathcal{I}\gamma_\mu|^k_j(\sqrt{g}T^{i\mu\nu})]h_{kl}[\mathcal{I}\gamma_\kappa|^l_j(\sqrt{g}T^{j\kappa\nu})]\} \\ \mathcal{I}\mathcal{L}_{AA} &= \tfrac{2}{2m^2}[\mathcal{R}\nabla_\mu|^k_i(\sqrt{g}T^{i\mu\nu})]h_{kl}[\mathcal{I}\nabla_\kappa|^l_j(\sqrt{g}T^{j\kappa\nu})]\end{aligned}$$

on one hand, and in the Riemann-like tensor,

$$\mathcal{R}R^k{}_{j\mu\nu} = [\mathcal{R}\nabla_\mu, \mathcal{R}\nabla_\nu]^k_j - [\mathcal{I}\nabla_\mu, \mathcal{I}\nabla_\nu]^k_j \quad (67)$$
$$\mathcal{I}R^k{}_{j\mu\nu} = [\mathcal{R}\nabla_\mu, \mathcal{I}\nabla_\nu]^k_j + [\mathcal{I}\nabla_\mu, \mathcal{R}\nabla_\nu]^k_j. \quad (68)$$

on the other. The connection to the imaginary part of $\breve{A}^a_\mu$ is more direct in Eq. (57) which yields,

$$\mathcal{R}F^a_{\mu\nu}(\breve{A}) = \tfrac{1}{2}\epsilon^{abc}d^{bj}d^c_k\mathcal{R}R^k{}_{j\mu\nu}, \quad (69)$$
$$\mathcal{I}F^a_{\mu\nu}(\breve{A}) = \tfrac{1}{2}\epsilon^{abc}d^{bj}d^c_k\mathcal{I}R^k{}_{j\mu\nu}, \quad (70)$$

Finally, the BF term becomes,

$$\mathcal{R}\mathcal{L}_{BF} = -\tfrac{1}{4}\sqrt{g}T^{j\mu\nu}\epsilon_{jmk}h^{lm}\mathcal{I}R^k{}_{l\mu\nu}, \quad (71)$$
$$\mathcal{I}\mathcal{L}_{BF} = +\tfrac{1}{4}\sqrt{g}T^{j\mu\nu}\epsilon_{jmk}h^{lm}\mathcal{R}R^k{}_{l\mu\nu}. \quad (72)$$

Summing up, at the complex saddle point of the $[dA]$ integration the emerging Euclidean $\mathcal{L}_{AA}$ and $\mathcal{L}_{BF}$ are both complex, whereas before they were real and purely imaginary, respectively. Both terms together determine the saddle point value $\breve{A}^a_\mu$. Therefore, they become coupled and cannot be considered separately anymore. This was already to be expected from the analysis in Minkowski space in Sect. II, where the matrix $\mathbb{M}^{ab}_{\mu\nu}$ combines T-odd and T-even contributions, which originate from $\mathcal{L}_{AA}$ and $\mathcal{L}_{BF}$, respectively. There the different contributions become entangled when the inverse $(\mathbb{M}^{-1})^{ab}_{\mu\nu}$ is calculated.

A. Weinberg–Salam model

Finally, let us reformulate the Weinberg–Salam model in geometric variables. We omit here the kinematic term $K(m)$ and the potential term $V(m^2)$ for the sake of brevity because they do not interfere with the calculations and can be reinstated at every time. The remaining terms of the classical action are

$$S := \int d^4x(\mathcal{L}^{\rm Abel}_{BB} + \mathcal{L}^{\rm Abel}_{BF} + \mathcal{L}_{BB} + \mathcal{L}_{BF} + \mathcal{L}_{AA}),$$
$$\mathcal{L}_{AA} := -\tfrac{m^2}{2}\mathbf{m}^{\underline{ab}}A^{\underline{a}}_\mu A^{\underline{b}}_\mu, \quad (73)$$
$$\mathcal{L}^{\rm Abel}_{BB} := -\tfrac{g^2}{4}B^0_{\mu\nu}B^0_{\mu\nu}, \quad (74)$$
$$\mathcal{L}^{\rm Abel}_{BF} := +\tfrac{i}{4}\epsilon^{\mu\nu\kappa\lambda}B^0_{\mu\nu}F^0_{\kappa\lambda}, \quad (75)$$

and $\mathcal{L}_{BB}$ as well as $\mathcal{L}_{BF}$ have been defined in Eqs. (40) and (41), respectively.

The saddle point conditions for the $[dA]$ integration with this action are given by

$$\tfrac{1}{2}\epsilon^{\kappa\lambda\mu\nu}D^{ab}_\mu(\breve{A})B^b_{\kappa\lambda} = +im^2\mathbf{m}^{a\underline{b}}A^{\underline{b}}_\nu, \quad (76)$$
$$\tfrac{1}{2}\epsilon^{\kappa\lambda\mu\nu}\partial_\mu B^0_{\kappa\lambda} = +im^2\mathbf{m}^{0\underline{b}}A^{\underline{b}}_\nu. \quad (77)$$

For the following it is convenient to use linear combinations of these equations, which are obtained by contraction with the eigenvectors $\mu^{\underline{a}}_l$ of the matrix $\mathbf{m}^{\underline{ab}}$—defined between Eqs. (37) and (38)—,

$$\tfrac{1}{2}\epsilon^{\kappa\lambda\mu\nu}[\mu^a_l D^{ab}_\mu(\breve{A})B^b_{\kappa\lambda} + \mu^0_l\partial_\mu B^0_{\kappa\lambda}] = im^2\mu^{\underline{a}}_l\mathbf{m}^{\underline{ab}}A^{\underline{b}}_\nu. \quad (78)$$

The non-Abelian term on the left-hand side can be rewritten using the results from the first part of Sect. III. The right-hand side may be expressed in terms of eigenvalues of the matrix $\mathbf{m}^{\underline{ab}}$. We find (no summation over l),

$$\mu^{\underline{a}}_l X^{\underline{a}\nu} = im^2 m_l a^l_\nu, \quad (79)$$

where

$$X^{\underline{a}\nu} := d^a_j\nabla_\mu|^j_k(\sqrt{g}T^{k\mu\nu}) + \tfrac{1}{2}\epsilon^{\kappa\lambda\mu\nu}\mu^0_l\partial_\mu B^0_{\kappa\lambda}. \quad (80)$$

The mass term can be decomposed in the eigenbasis of $\mathbf{m}^{\underline{ab}}$ as well and, subsequently, be formulated in terms of the geometric variables,

$$\begin{aligned}\mathcal{L}_{AA} &= -\tfrac{m^2}{2}\textstyle\sum_l m_l a^l_\mu a^l_\mu = \\ &= \tfrac{1}{2m^2}(\bar{\mathbf{m}}^{-1})^{\underline{ab}}X^{\underline{a}\nu}X^{\underline{b}\nu},\end{aligned} \quad (81)$$

where

$$(\bar{\mathbf{m}}^{-1})^{\underline{ab}} := \textstyle\sum_l^{\forall m_l\neq 0} m_l{}^{-1}\mu^{\underline{a}}_l l\mu^{\underline{b}}_l. \quad (82)$$

With the help of these relations and the results from the beginning of Sect. III we are now in the position to express the classical action in geometric variables: The mass term is given in the previous expression. It describes a Gaussian distribution of a composite current. The components of the current are superpositions of Abelian and non-Abelian contributions. This mixture is caused by the symmetry breaking pattern $SU(2)_L \times U(1)_Y \to U(1)_{\rm em}$ which leaves unbroken $U(1)_{\rm em}$ and not the $U(1)_Y$ which is a symmetry in the unbroken phase. The Abelian antisymmetric fields $B^0_{\mu\nu}$ in $\mathcal{L}^{\rm Abel}_{BB}$ are gauge invariant and we leave $\mathcal{L}^{\rm Abel}_{BB}$ as defined in Eq. (74). In geometric variables $\mathcal{L}_{BB}$ is given

by Eq. (48) and $\mathcal{L}_{BF}$ by Eq. (58). At the end the kinetic term $K(m)$ and the potential term $V(m^2)$ should be reinstated.

Additional contributions from fluctuations give rise to an addend (on the level of the Lagrangian) proportional to $\frac{1}{2}\ln\det\mathbb{m}$, where $\mathbb{m}$ can be expressed in the new variables, $\mathbb{m}^{jk}_{\mu\nu} = f^{abc} d^a_l \mu^b_j \mu^c_k \sqrt{\mathrm{g}} T^{l\mu\nu} - m^2 \sum_l m_l \delta^{lj}\delta^{kl} g_{\mu\nu}$.

Repeating the entire calculation not in unitary gauge, but with explicit gauge scalars Φ, yields exactly the same result because the mass term and the saddle point condition change in unison, such that Eq. (79) is obtained again. This has already been demonstrated explicitly for a massive Yang–Mills theory just before Sect. III A.

IV. SUMMARY

We have discussed the formulation of massive gauge field theories in terms of antisymmetric tensor fields (Sect. II) and of geometric variables (Sect. III). The description in terms of an antisymmetric tensor field $B^a_{\mu\nu}$ has the advantage that it transforms homogeneously under gauge transformations, whereas the usual gauge field A^a_μ transforms inhomogeneously, which complicates a gauge-independent treatment of massive gauge field theories. In fact, the (Stückelberg-like) degrees of freedom needed for a gauge-invariant formulation in terms of a Yang–Mills connections are directly absorbed in the antisymmetric tensor fields. No scalar field is required in order to construct a gauge invariant massive theory in terms of the new variables. After recapitulating the massless case in Sect. IIA, we have treated the massive setting in Sect. IIB. After the fixed mass case, at the beginning of Sect. IIB, this section encompasses also a position dependent mass (Sect. IIB1), that is the Higgs degree of freedom, and a non-diagonal mass term (Sect. IIB2). This is required for describing the Weinberg–Salam model. In this context, we have identified the degrees of freedom which represent the different electroweak gauge bosons in the $B^a_{\mu\nu}$ representation by a gauge-invariant eigenvector decomposition.

The Abelian section (App. A) serves as basis for an easier understanding of some issues arising in the non-Abelian case, like for example vanishing conserved currents. In that section we also address the massless limits of propagators in the A_μ and $B_{\mu\nu}$ representations, respectively. We notice that while the limit is ill-defined for the A_μ fields it is well-defined for the $B_{\mu\nu}$ fields. That is due to the consistent treatment of gauge degrees of freedom in the latter case.

In Sect. III we continue with a description of massive gauge field theories in terms of geometric variables in four space-time dimensions and for two colours. Thereby we can eliminate the remaining degrees of freedom which are still encoded in the $B^a_{\mu\nu}$ fields. After deriving the expressions for a fixed mass and in the presence of the Higgs degree of freedom, respectively, we also investigate the Weinberg–Salam model (Sect. III A).

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APPENDIX A: ABELIAN

1. Massless

The partition function of an Abelian gauge field theory without fermions is given by

$$P := \int [dA] \exp\{i \textstyle\int d^4x \mathcal{L}\} \tag{A1}$$

with the Lagrangian density

$$\mathcal{L} = \mathcal{L}_0 := -\tfrac{1}{4g^2} F_{\mu\nu}F^{\mu\nu} \tag{A2}$$

and the field tensor

$$F_{\mu\nu} := \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{A3}$$

g stands for the coupling constant. The transition to the first-order formalism can be performed just like in the non-Abelian case, which is treated in the main body of the paper. We find the partition function,

$$\begin{aligned} P &= \int [dA][dB] \times \\ &\quad \times \exp\{i \textstyle\int d^4x [-\frac{1}{2}\tilde{F}_{\mu\nu}B^{\mu\nu} - \frac{g^2}{4} B_{\mu\nu}B^{\mu\nu}]\}. \end{aligned} \tag{A4}$$

Here the antisymmetric tensor field $B_{\mu\nu}$, like the field tensor $F_{\mu\nu}$, is gauge invariant. The classical equations of motion are given by

$$\partial_\mu \tilde{B}^{\mu\nu} = 0 \quad \text{and} \quad g^2 B_{\mu\nu} = -\tilde{F}_{\mu\nu}, \tag{A5}$$

which after elimination of $B_{\mu\nu}$ reproduce the Maxwell equations one would obtain from Eq. (A2). Now we can formally integrate out the gauge field A_μ. As no gauge is fixed by the BF term because the Abelian field tensor $F_{\mu\nu}$ is gauge invariant this gives rise to a functional δ distribution. This constrains the allowed field configurations to those for which the conserved current $\partial_\mu \tilde{B}^{\mu\nu}$ vanishes,

$$P \cong \int [dB] \delta(\partial_\mu \tilde{B}^{\mu\nu}) \exp\{i \textstyle\int d^4x [-\frac{g^2}{4} B_{\mu\nu}B^{\mu\nu}]\}. \tag{A6}$$

2. Massive

In the massive case the Lagrangian density becomes $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_m$, where $\mathcal{L}_m := \frac{m^2}{2} A_\mu A^\mu$. First, we here

repeat some steps carried out above in the non-Abelian case: We can directly write down the partition function in unitary gauge. Regauging like in Eq. (18) leads to

$$P = \int [dA]'[dU] \exp(i\textstyle\int d^4x\{-\frac{1}{4g^2}F_{\mu\nu}F^{\mu\nu} + \\ +\frac{m^2}{2}[A_\mu - iU^\dagger(\partial_\mu U)][A^\mu - iU^\dagger(\partial^\mu U)]\}). \quad (A7)$$

The corresponding gauge-invariant Lagrangian then reads,

$$\mathcal{L}_{\rm cl} := -\tfrac{1}{4g^2}F_{\mu\nu}F^{\mu\nu} + \tfrac{m^2}{2}(D_\mu\Phi)^\dagger(D^\mu\Phi), \quad (A8)$$

with the constraint $\Phi^\dagger\Phi \stackrel{!}{=} 1$. Constructing a partition function in the first-order formalism from the previous Lagrangian yields,

$$P \cong \int [dA][d\Phi][dB] \times \\ \times \exp(i\textstyle\int d^4x\{-\frac{1}{2}B_{\mu\nu}\tilde{F}^{\mu\nu} - \frac{g^2}{4}B_{\mu\nu}B^{\mu\nu} + \\ +\frac{m^2}{2}[A_\mu - i\Phi(\partial_\mu\Phi^\dagger)][A^\mu - i\Phi(\partial^\mu\Phi^\dagger)]\}). \quad (A9)$$

The Φ fields can be absorbed entirely in a gauge-transformation of the gauge field A_μ. The integration over Φ decouples. This can also be seen by putting the parametrisation $\Phi = e^{-i\theta}$ into the previous equation and carrying out the $[dA]$ integration,

$$P \cong \int [dB][d\theta] \exp\{i\textstyle\int d^4x[-\frac{g^2}{4}B_{\mu\nu}B^{\mu\nu} - \\ -\frac{1}{2m^2}(\partial_\kappa\tilde{B}^{\kappa\mu})g_{\mu\nu}(\partial_\lambda\tilde{B}^{\lambda\nu}) - (\partial_\mu\theta)(\partial_\kappa B^{\kappa\mu})]\}. \quad (A10)$$

The only θ dependent term in the exponent is a total derivative and drops out, leading to a factorisation of the θ integral.

A third way which yields the same final result, starts by integrating out the θ field first. This gives a transverse mass term $\sim A^\mu(g_{\mu\nu} - \frac{\partial_\mu\partial_\nu}{\Box})A^\nu$. Integration over A_μ then leads to the same result as before.

Instead of a vanishing current $\partial_\mu\tilde{B}^{\mu\nu}$ like in the massless case, in the massive case the current has a Gaussian distribution. The distribution's width is proportional to the mass of the gauge boson.

$m \to 0$ limit

In the gauge-field representation the massless limit for the classical actions discussed above are smooth. In terms of the $B_{\mu\nu}$ field the mass m ends up in the denominator of the corresponding term in the action. Together with the m dependent normalisation factors arising form the integrations over the gauge-field in the course of the derivation of the $B_{\mu\nu}$ representation, however, the limit $m \to 0$ still yields the $m = 0$ result for the partition function (A6).

Still, it is known that the perturbative propagator for a massive photon is ill-defined if the mass goes to zero: Variation of the exponent of the Abelian massive partition function in unitary gauge with respect to A_κ and A_λ gives the inverse propagator for the gauge fields,

$$(G^{-1})^{\kappa\lambda} = [(p^2 - m^2_{\rm phys})g^{\kappa\lambda} - p^\kappa p^\lambda], \quad (A11)$$

which here is already transformed to momentum space. The corresponding equation of motion,

$$(G^{-1})^{\kappa\lambda}G_{\lambda\mu} \stackrel{!}{=} g^\kappa_\mu, \quad (A12)$$

is solved by

$$G_{\lambda\mu} = \frac{g_{\lambda\mu}}{p^2 - m^2_{\rm phys}} - \frac{1}{m^2_{\rm phys}}\frac{p_\lambda p_\mu}{p^2 - m^2_{\rm phys}}, \quad (A13)$$

with boundary conditions (an ϵ prescription) to be specified and $m_{\rm phys} := mg$. This propagator diverges in the limit $m \to 0$.

In the representation based on the antisymmetric tensor fields, variation of the exponent of the partition function (A10) with respect to the fields $\tilde{B}_{\mu\nu}$ and $\tilde{B}_{\kappa\lambda}$ yields the inverse propagator

$$(G^{-1})^{\mu\nu|\kappa\lambda} = g^{\mu\kappa}g^{\nu\lambda} - g^{\nu\kappa}g^{\mu\lambda} + \\ +m^{-2}_{\rm phys}(\partial^\mu\partial^\kappa g^{\nu\lambda} - \partial^\nu\partial^\kappa g^{\mu\lambda} - \\ -\partial^\mu\partial^\lambda g^{\nu\kappa} + \partial^\nu\partial^\lambda g^{\mu\kappa}), \quad (A14)$$

already expressed in momentum space. Variation with respect to $\tilde{B}_{\mu\nu}$ instead of $B_{\mu\nu}$ corresponds only to a reshuffling of the Lorentz indices and gives an equivalent description. The antisymmetric structure of the inverse propagator is due to the antisymmetry of $\tilde{B}_{\mu\nu}$. The equation of motion is then given by

$$(G^{-1})^{\mu\nu|\kappa\lambda}G_{\kappa\lambda|\rho\sigma} \stackrel{!}{=} g^\mu_\rho g^\nu_\sigma - g^\mu_\sigma g^\nu_\rho \quad (A15)$$

and solved by

$$2G_{\kappa\lambda|\rho\sigma} = (g_{\kappa\rho}g_{\lambda\sigma} - g_{\kappa\sigma}g_{\lambda\rho}) - \frac{1}{p^2 - m^2_{\rm phys}} \times \\ \times(p_\kappa p_\rho g_{\lambda\sigma} - p_\kappa p_\sigma g_{\lambda\rho} - \\ p_\lambda p_\rho g_{\kappa\sigma} + p_\lambda p_\sigma g_{\kappa\rho}). \quad (A16)$$

Here we observe that the limit $m \to 0$ is well-defined,

$$2G_{\kappa\lambda|\rho\sigma} \xrightarrow{m\to 0} g_{\kappa\rho}g_{\lambda\sigma} - g_{\kappa\sigma}g_{\lambda\rho} - \\ -\frac{1}{p^2}(p_\kappa p_\rho g_{\lambda\sigma} - p_\kappa p_\sigma g_{\lambda\rho} - \\ -p_\lambda p_\rho g_{\kappa\sigma} + p_\lambda p_\sigma g_{\kappa\rho}). \quad (A17)$$

This is due to the consistent treatment of the gauge degrees of freedom in the second approach.

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[26] We do not discuss here odd dimensional cases, where mass can be created topologically by means of Chern–Simons terms [19].

[27] Again in an odd number of space-time dimensions (2+1) Karabali and Nair [20] also together with Kim [21] have carried out a gauge invariant analysis of non-Abelian gauge field theories. As pointed out by Freidel et al. [22] this approach appears to be extendable to 3+1 dimensions making use of the so-called "corner variables" [23].

[28] The functional integral over a gauge field is ill-defined as long as no gauge is fixed. We have to keep this fact in mind at all times and will discuss it in detail when a field is really integrated out.

[29] Due to the possible occurrence of what is known as Gribov copies [24] this method might not achieve a unique splitting of the two parts of the integration. For our illustrative purposes, however, this is not important.

[30] For *three* colours and four space-time dimensions there exists a treatment of the massless setting due to Lunev [25], different from the one which here is extended to the massive case.

[31] One possible representation is ${\eta^i}_{mn} = {\epsilon^a}_{mn}$, $\eta^i_{4n} = -\delta^a_n$, $\eta^i_{m4} = -\delta^a_m$, and ${\eta^i}_{44} = 0$, where $m, n \in \{1; 2; 3\}$.